Multi-dimensional Sampling in Fan Beam Tomography

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Abstract

In this paper we explore a new sampling scheme proposed by S.H. Izen, D.P. Rohler and Sastry K.L.A. for fan beam tomography. This method exploits the reflection property of the fan beam transform via multi-dimensional sampling theory and is more efficient than the standard scheme in the sense that half as many measurements are needed to obtain the same resolution. We depart from the work of Izen et al. by employing Faridani's generalized sampling expansion.

Purpose

- Explain the process used by Izen et al. to overcome the so-called third-generation problem.
- Illustrate how this can be combined with Faridani's multi-dimensional sampling theory to eliminate some of the restrictive hypothesis of Izen et al..

OUTLINE

1) Introductory Material
2) Sampling Theory
3) Union of Lattices
4) Simple Example
5) Results
6) Conclusion

INTRODUCTION

What is Tomography?

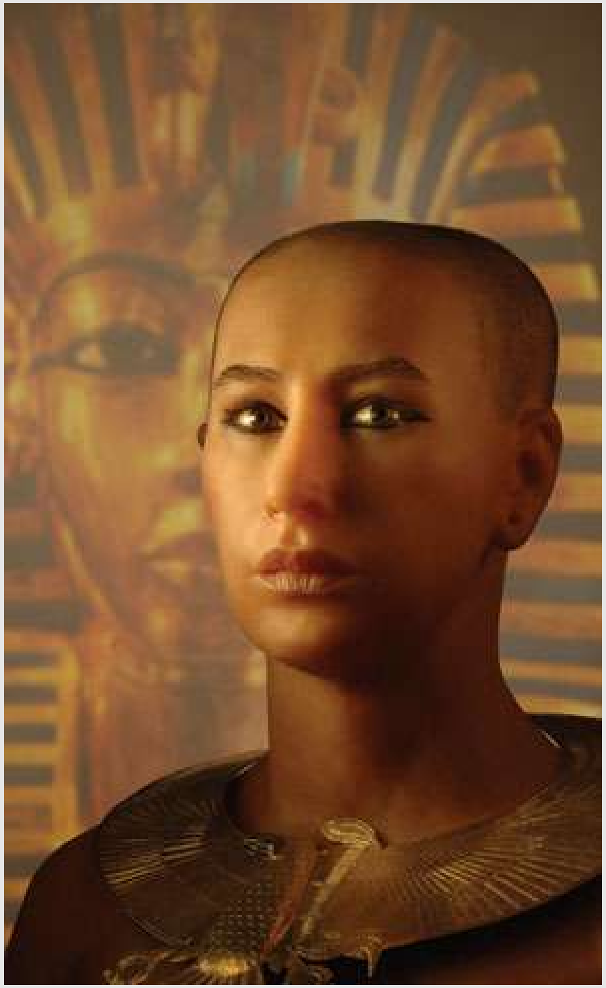
- A non-invasive way to see inside objects
- Comes from the Greek word for *slice*
- It is the “T” in CT-scan

What happens in a CT-scan?

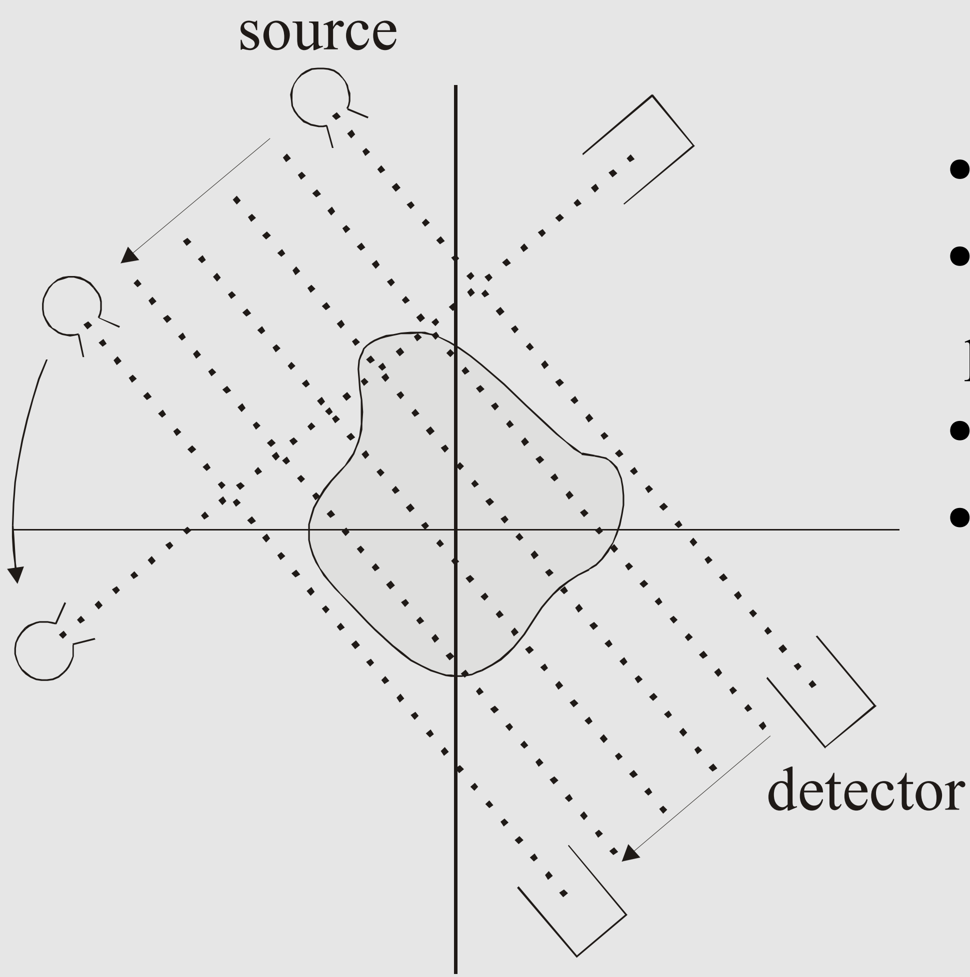
- Beams of x-rays are fired through the object
- Some rays are absorbed or scattered
- Detectors measure the final strength of the beam
- The data are combined in a computerized algorithm to produce an image of the interior

King Tut

On January 5, 2005 a CT scan was made on King Tutankhamun's mummy. This gave foresic scientists enough information to make a fairly accurate reconstruction of the living Pharoah.

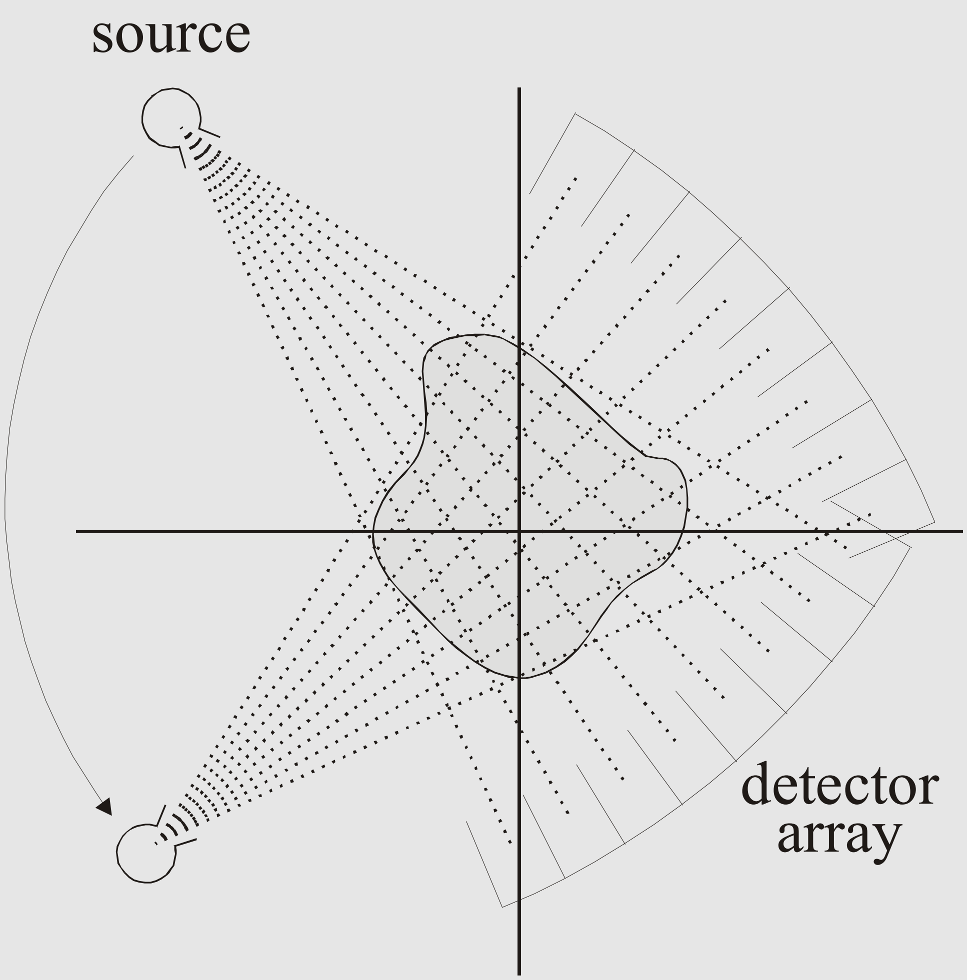


Parallel Beam Scanning Geometry



- One source, one detector
- The source and detector move in parallel and then rotate
- Are fairly slow
- Used in 1st-generation scanners

Fan Beam Scanning Geometry



- Single source, multiple detectors
- Source and detector array rotate continuously—no translating
- Much faster than parallel beam
- Used in 3rd-generation scanners

3^{rd} Generation Problem

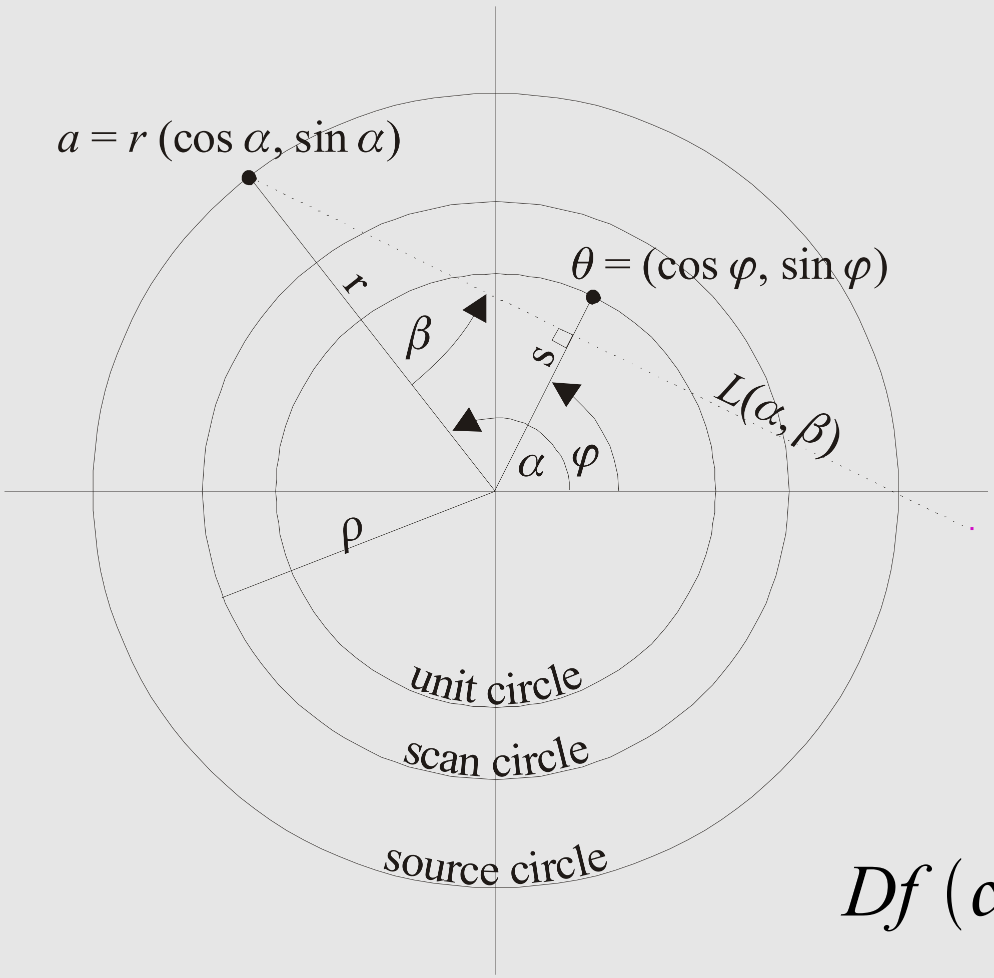
The detector array of a third-generation scanner is an efficient way to capture most of the x-rays. However, it reduces the resolution to half of what is theoretically possible.

Line Integrals

Each recorded measurement gives us an integral along the path of the x-ray beam. If we can invert the integral we can recover the density function $f(x)$ which describes the interior of the object.

$$\ln\left(\frac{I_o}{I}\right) = \int_L f(x)\,dx$$

The 2D Transforms



The Radon Transform

$$Rf(\varphi, s) = \int_{L(\varphi, s)} f(x)\,dx$$

The Fan Beam Transform

$$Df(\alpha, \beta) = \int_{L(\alpha, \beta)} f(x)\,dx$$

Df is related to *Rf* by

$$Df(\alpha, \beta) = Rf(\alpha + \beta - \pi/2, r\sin\beta)$$

Reflection Property

Because of the symmetry of the problem, each measurement can be used twice. We will use this property later to increase the resolution of the reconstructed image.

$$Df(\alpha, \beta) = Df(\pi + \alpha + 2\beta, -\beta)$$

SAMPLING THEORY

Sampling Theory

Sampling theory allows us to answer two important questions:

- Which measurements should we take?
- How many do we need to get a good picture?

Fourier Analysis

Fourier Transform

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{i\langle x, \xi \rangle} dx$$

Inverse Fourier Transform

$$\tilde{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-i\langle x, \xi \rangle} dx$$

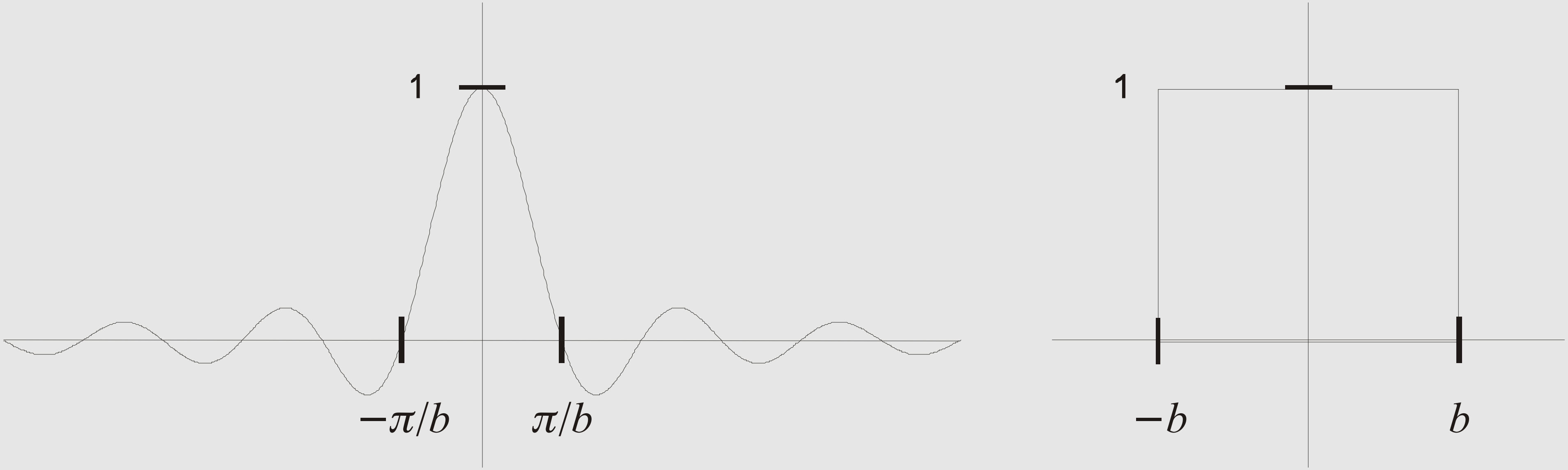
Fourier Inversion Formula

$$\hat{\tilde{f}} = \tilde{\hat{f}} = f$$

Bandlimited Functions

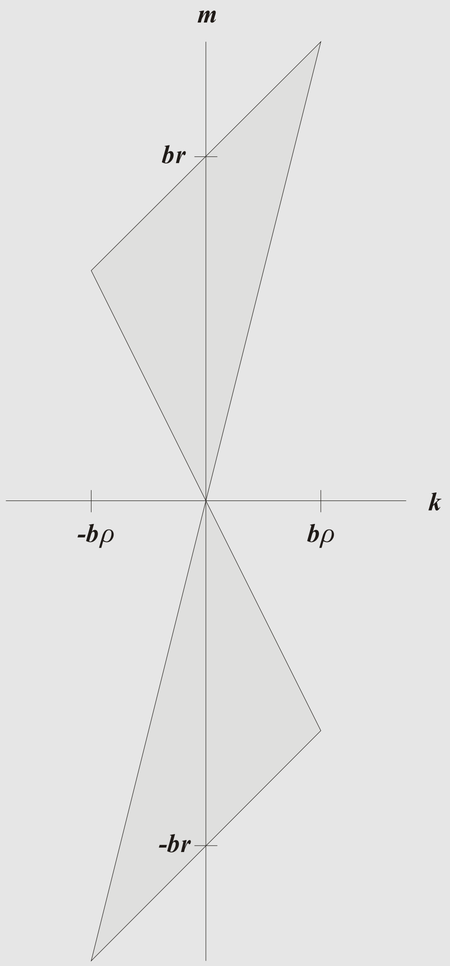
A function is bandlimited with bandwidth *b*, or *b-bandlimited*, if its Fourier transform is zero outside of a region of radius *b*. It is *essentially b*-bandlimited if its Fourier transform is small outside of that region.

b-bandlimited functions cannot represent details smaller than 2π/b.



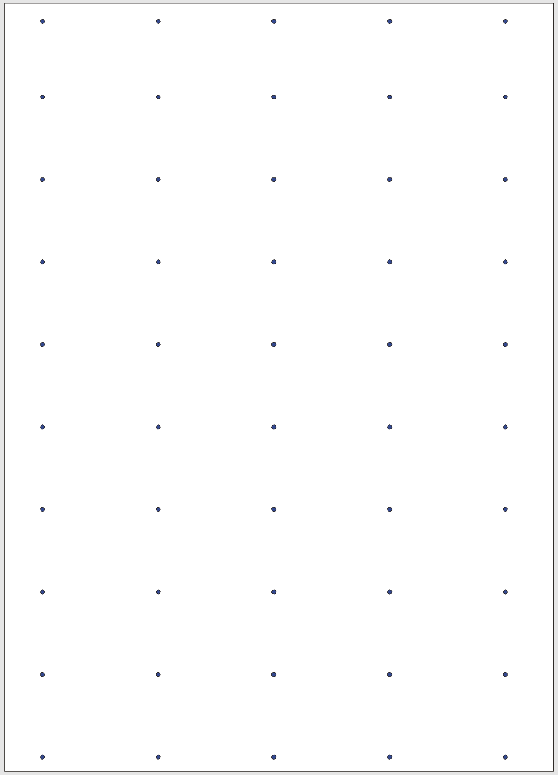
Essential support of $\widehat{Df}$

If f is essentially b-bandlimited, r is the radius of the source circle and ρ is the radius of the scan circle, then the essential support, or bandregion K of $\widehat{Df}$ has the following shape.



Lattices

- A lattice is a multi-dimensional grid
- To effectively sample a function on a grid the grid must be sufficiently dense
- It must also match the periodicity of the function



One way to create a lattice is to multiply a matrix *W* by the integer lattice, ie.

$$L_W = W \cdot \mathbb{Z}^n$$

Each lattice has a dual lattice

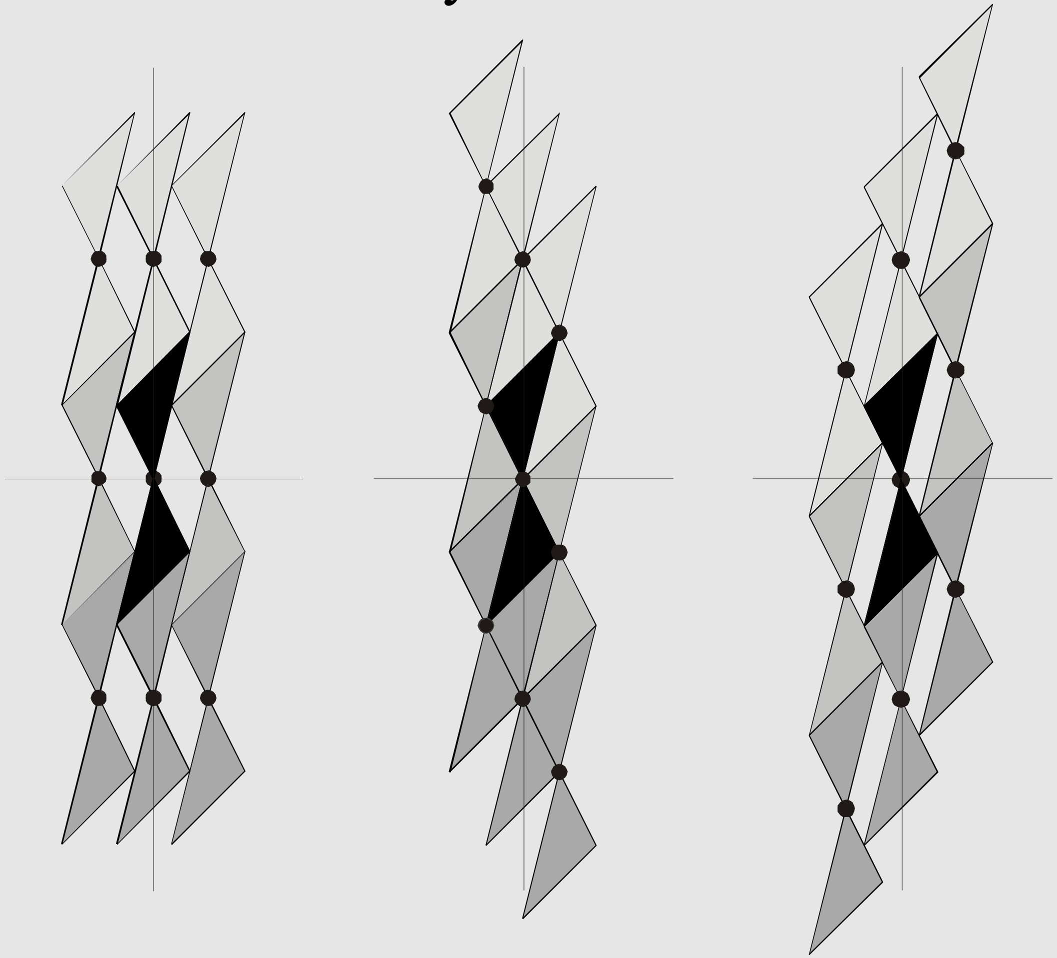
$$L_W^{\perp} = 2\pi W^{-T} \cdot \mathbb{Z}^n$$

Translates of K

A lattice is sufficiently dense if shifts of K by elements of the dual lattice do not overlap K.

<u>From left to right</u>

- standard lattice
- efficient lattice
- reflected lattice



K is black. Dots indicate elements of the dual lattice.

Sampling Conditions

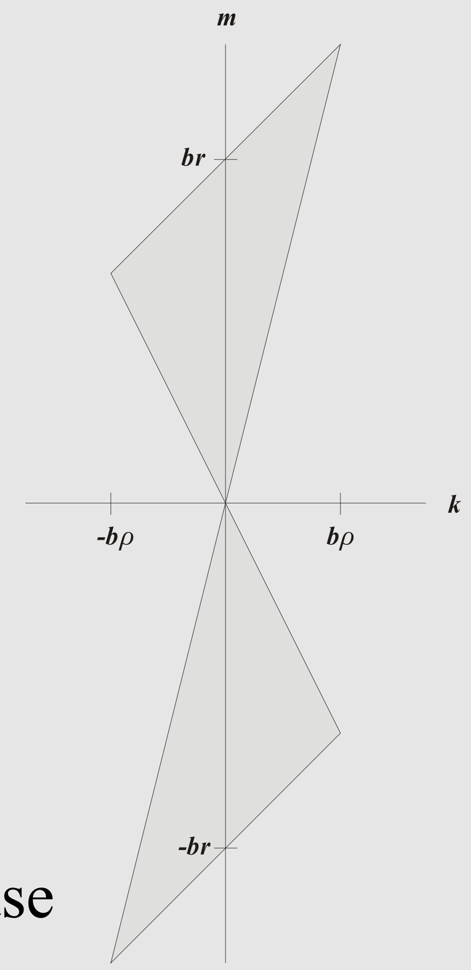
To ensure that translates of K do not overlap the standard scheme requires the matrix L_W

$$L_W = \begin{pmatrix} \Delta\alpha & 0 \\ 0 & \Delta\beta \end{pmatrix}$$

to satisfy the sampling conditions

$$\Delta\alpha = \frac{2\pi}{p} \le \frac{r+\rho}{\rho} \cdot \frac{\pi}{br}\ , \qquad \Delta\beta = \frac{\pi}{q} \le \frac{\pi}{br}$$

Violating these conditions leads to overlap which can cause artifacts, or false images, in the reconstructed image.



Shift-Convexity

The algorithm developed by Izen et al. requires the set K to be *shift-convex.*

Roughly speaking, this means that not only should shifts by elements of the dual lattice not overlap K, but that additional shifts by some lattice L_P do not reintroduce overlap.

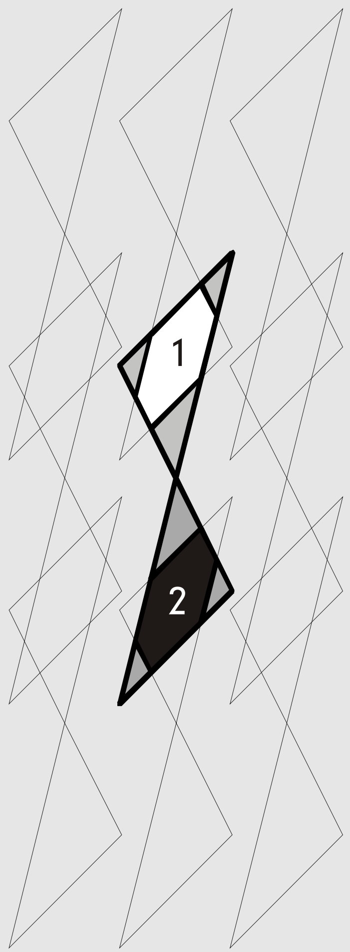
Here (a) is not shift-convex while (b) would be.

(a) (b)

Decomposition of K

When a lattice is not sufficiently dense, shifts of K by elements of the dual lattice will overlap K. Grouping together the regions of K which are covered by the same translates decomposes K into mutually disjoint sets.

Here K is decomposed into three sets: the White, Black and Gray regions.



Faridani's Multi-Dimensional Sampling Theorem

If K is the essential bandregion of f and can be decomposed into the union of mutually disjoint measurable sets; if W is a feasable (n_1+n_2, n_1+n_2) dimensional matrix; and if $a_1 \ldots a_m$ are real n_1+n_2-dimensional vectors such that there exists coefficients b_r^l satisfying

$$\sum_{r=1}^{m} b_r^l = 1 \qquad \sum_{r=1}^{m} b_r^l e^{-2\pi i \langle W^{-1} a_r, k_{i,j} \rangle} = 0$$

Then

$$f(x) \approx \sum_{r=1}^{m} \sum_{k \in A_W} f(a_r + Wk)\, g_r(x - a_r - Wk)$$

where

$$g_r(y) = (2\pi)^{-(n_1+n_2)/2} |det\, W| \sum_{l=1}^{L} b_r^l \widetilde{\chi_{K_l}}(y)$$

UNION OF LATTICES

Detector Shift

Most modern CT scanners allow the gantry to be shifted off-center by a fraction δ of a detector width.

Standard Lattice

When a detector shift is used with the standard scheme we obtain the original lattice $L_O=\{\alpha_j, \beta_l\}$ where

$$\alpha_j = j\,\Delta\alpha, \qquad \beta_l = (l+\delta)\,\Delta\beta,$$

$$\Delta\alpha = \frac{2\pi}{p}, \qquad \Delta\beta = \frac{\pi}{q},$$

$$j, l, p, q \in \mathbb{Z}$$

Reflected Lattice

Using the reflection property of *Df* we obtain the corresponding reflected samples $L_R=\{\alpha'_{j,l}, \beta'_l\}$ where

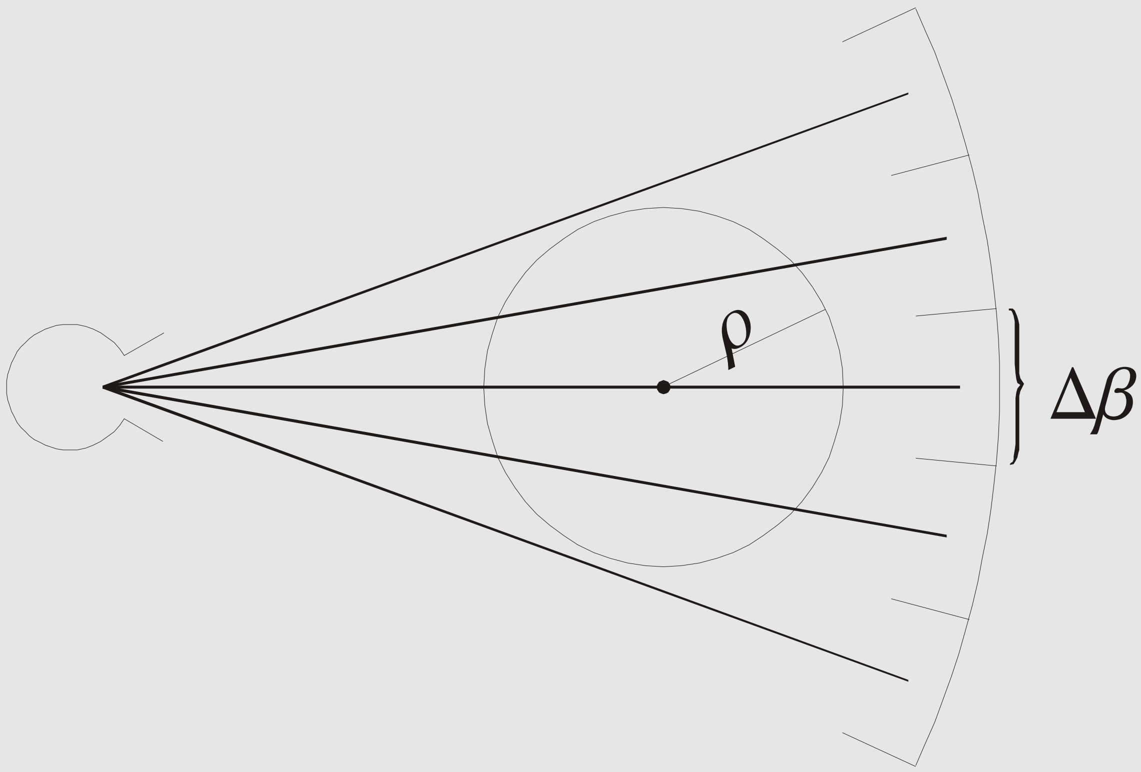
$$\alpha'_{j,l}=\pi+j\Delta\alpha+2(l+\delta)\Delta\beta, \qquad \beta_l=-(l+\delta)\Delta\beta,$$

$$\Delta\alpha=\frac{2\pi}{p}, \qquad \Delta\beta=\frac{\pi}{q},$$

$$j, l, p, q\in\mathbb{Z}$$

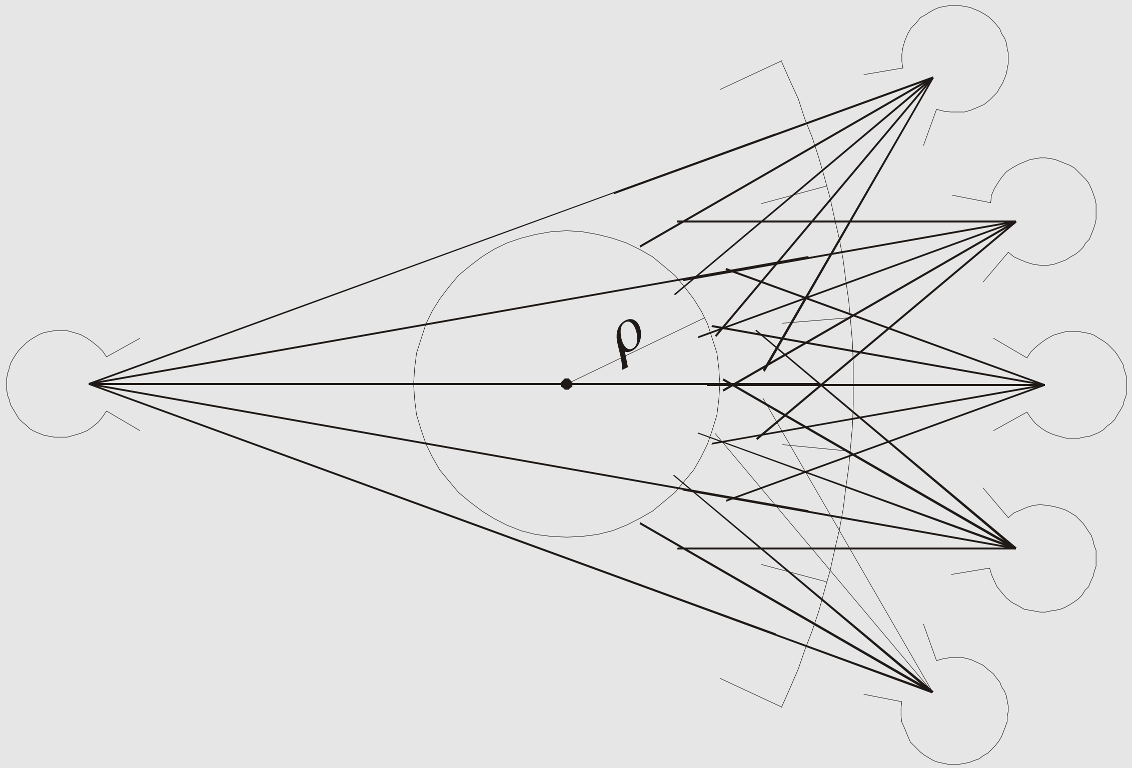
No Offset...

$\delta = 0$



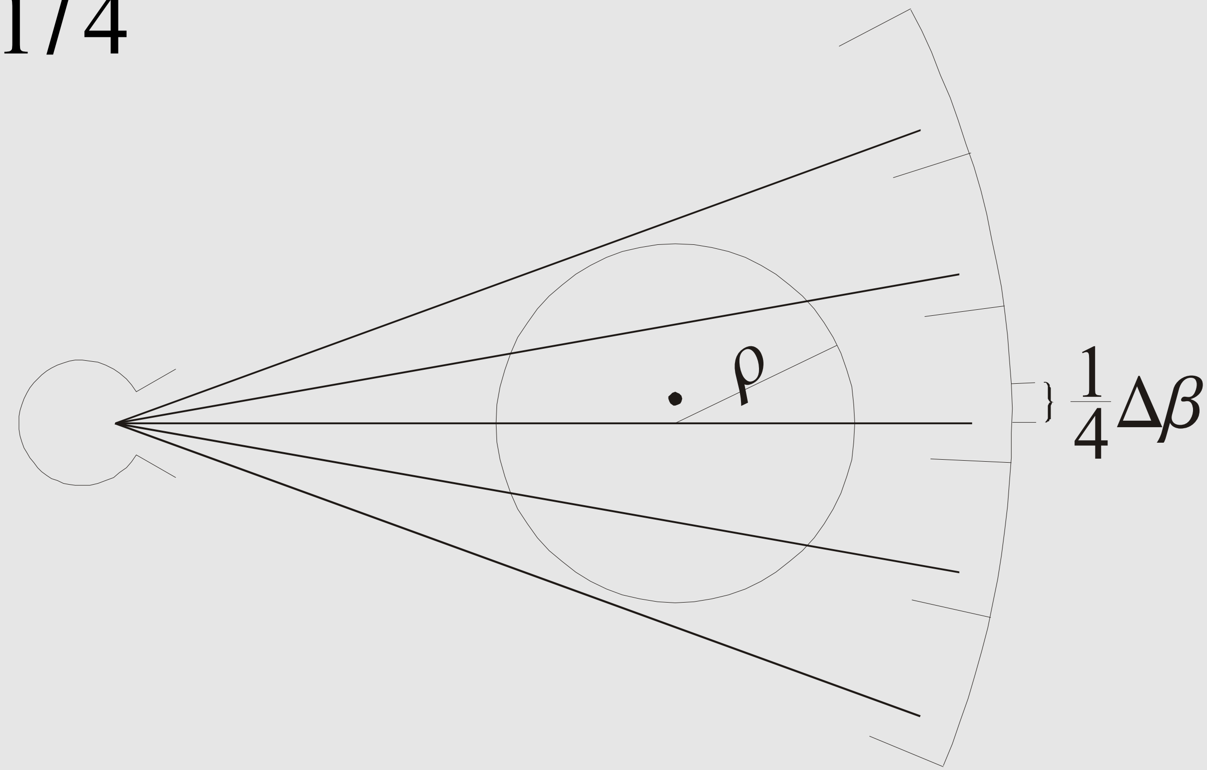
...may lead to duplication

$\delta = 0$



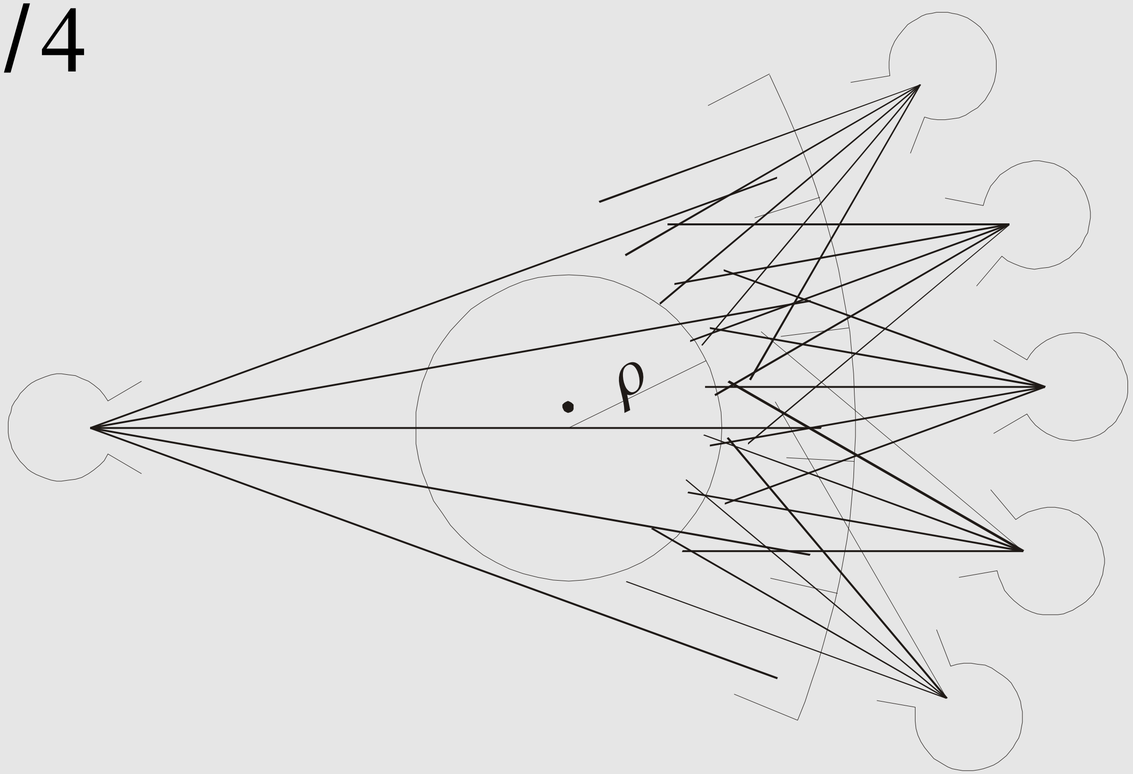
¼ detector width offset...

$\delta = 1/4$



...eliminates duplication

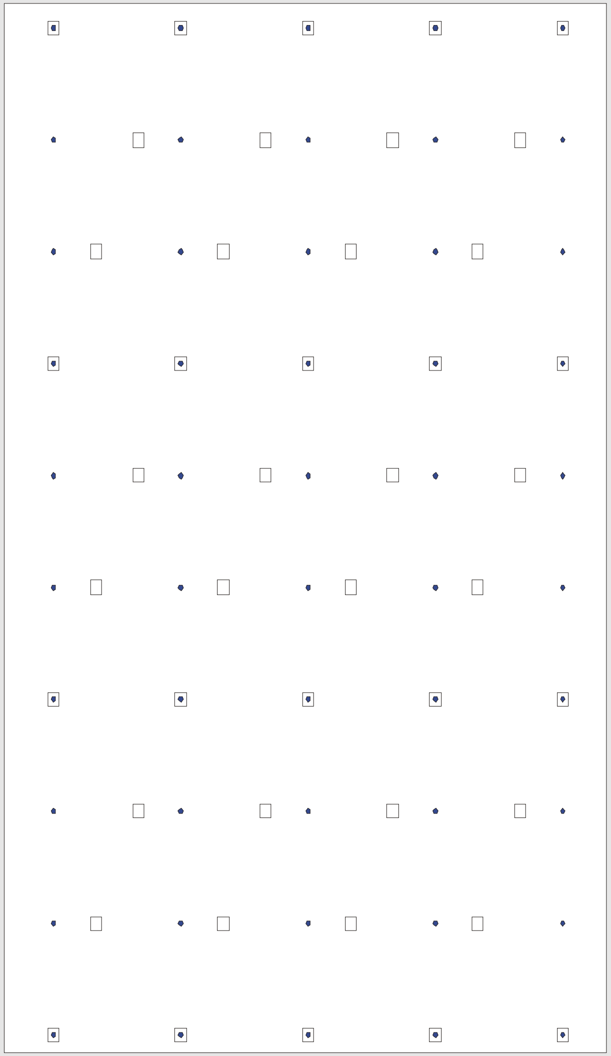
$\delta = 1/4$

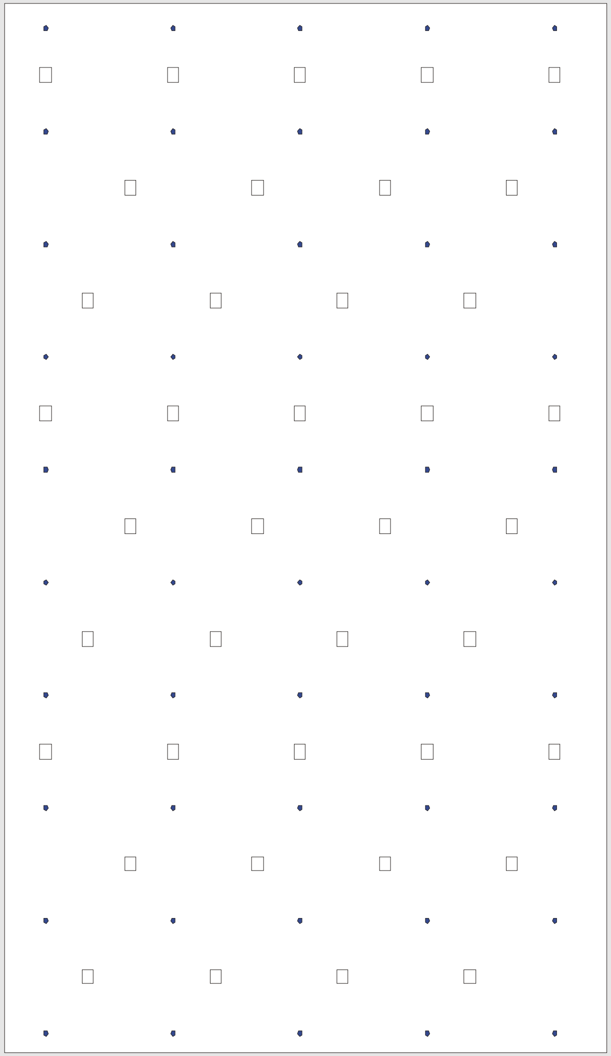


Interlacing L_O and L_R

With no detector offset (left) there is some duplication between L_O (dots) and L_R (squares).

However, with a shift of ¼ of a detector width (right) the two lattices interlace nicely.





Increased Data

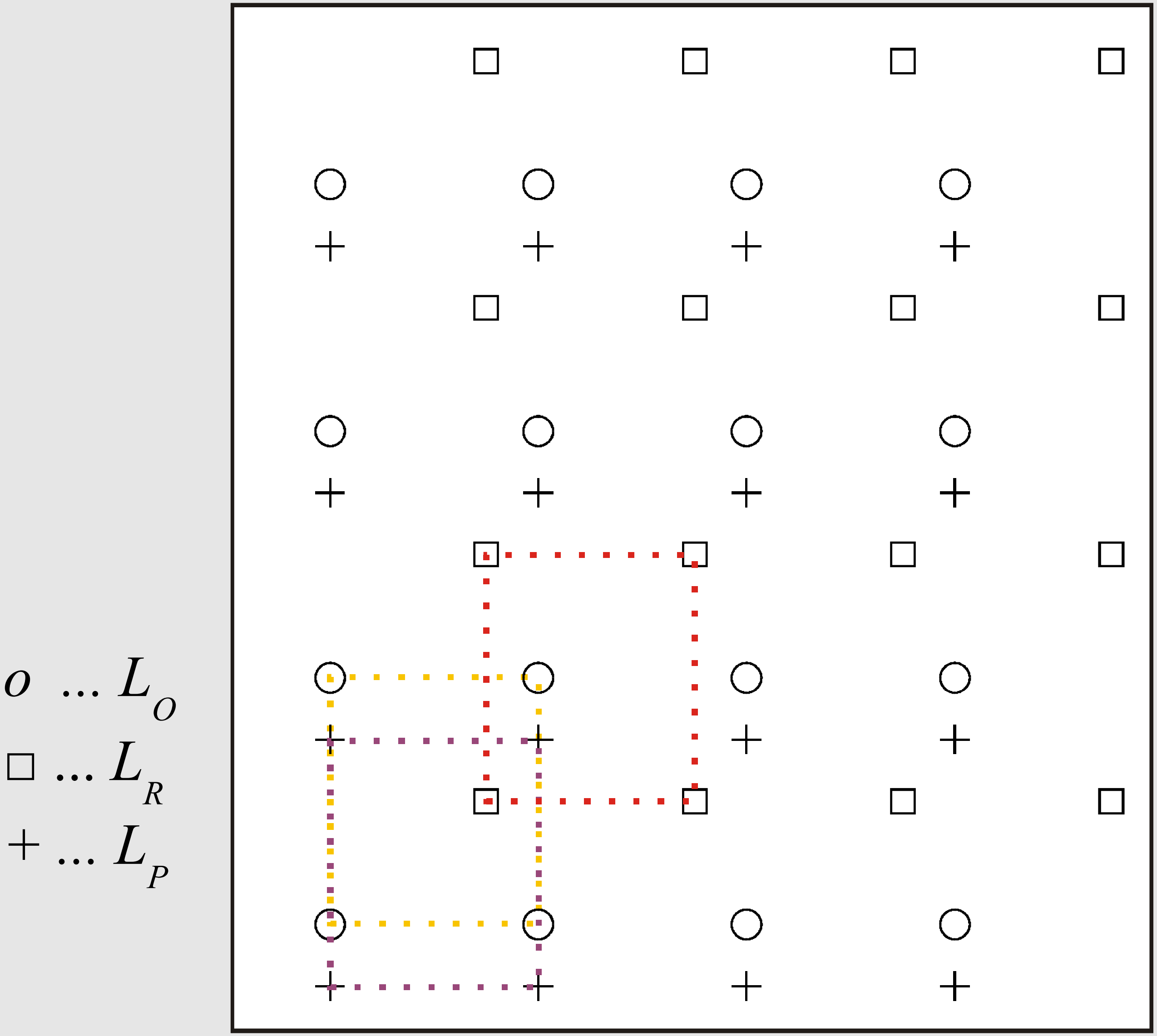
With a 1/4-detector width shift opposite views do not duplicate one another. Thus, the reflected samples are “extra” data and can be used to double the number of measurements.

In parallel beam scanning the original and reflected points line up on a single rectangular lattice. Such is not the case with fan beam scanning.

$L_O \cup L_R$

The central concept of the work of Izen et al. was the viewing of $L_O \cup L_R$ with a shift of ¼ detector width as the union of $2q'$ shifted copies of a single rectangular lattice L_P.

The case $q'=1$ is shown.



Union Theorem

$$L_O \cup L_R = \bigcup_{j=0}^{2q'} L_P + a_j$$

with

$$L_P = \begin{pmatrix} \Delta\alpha & 0 \\ 0 & q'\Delta\beta \end{pmatrix}, \qquad q' = \frac{q}{GCD(p,q)},$$

and , for $j = 0 \ldots q'$-1

$$a_j = \begin{pmatrix} 0 \\ (j+1/4)\Delta\beta \end{pmatrix}, \quad a_{j+q'} = \begin{pmatrix} \pi + \left(j + \frac{1}{4}\right)\frac{p}{q}\Delta\alpha \\ -(j+1/4)\Delta\beta \end{pmatrix}$$

A Simple Case

When p is an integer multiple of q then $q' = 1$ and

$$L_O \cup L_R = \left(L_P + a_1\right) \cup \left(L_P + a_2\right)$$

where

$$a_1 = \begin{pmatrix} 0 \\ \Delta\beta/4 \end{pmatrix} \qquad a_2 = \begin{pmatrix} \pi + p\,\Delta\alpha/(2q) \\ -\Delta\beta/4 \end{pmatrix}$$

RESULTS

The 3rd Generation Problem

The detectors in a 3rd generation scanner are capable of measuring details as small as $2\pi/2b$ but are limited to $2\pi/b$ because of their configuration. To reconstruct with a resolution of $2\pi/2b$ we would need to double both the source and the detector densities. Since the number of detectors is fixed we are unable to achieve a resolution of $2\pi/2b$ with the standard scheme.

Solving the 3^{rd} Generation Problem

Suppose we double the number of source locations (which poses no problem), use a 1/4-detector width shift, and compute the reflected data.

We can then use the theorem of Izen et al. to view the original and reflected data as the union of shifted copies of a single lattice.

Finally, Faridani's multidimensional sampling theorem can be used to interpolate the data which is missing with respect to the standard lattice.

Once that data is interpolated, any reconstruction algorithm can be used to reconstruct at a resolution of $2\pi/2b$, which corresponds to a bandwith of $2b$.

A Simple Experiment

We followed this process for the simple case of $q' = 1$ which, with equality in the sampling conditions, occurs when $\rho/r = 1/3$. The set K was as described before but with $2b$ in place of b.

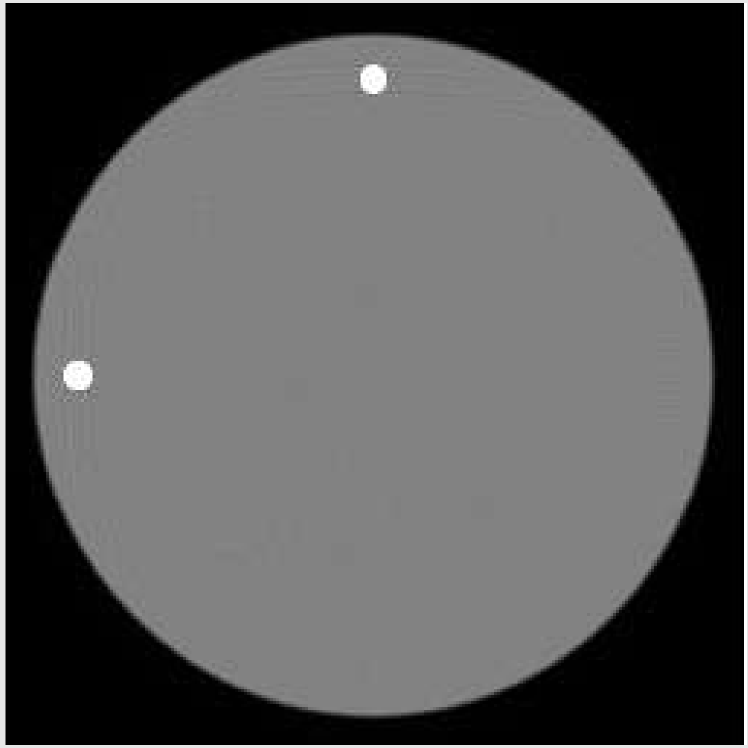
We simulated the data for two different phantoms on a standard lattice with a 1/4-detector shift with a source density corresponding to $2b$ but a detector density corresponding to a bandwidth of just b.

The lattice L_P and the shifts a_1 and a_2 computed and found to satisfy the requirements of Faridani's multi-dimensional sampling theory. Thus, we were able to interpolate the missing data to increase the detector density to the $2b$ level.

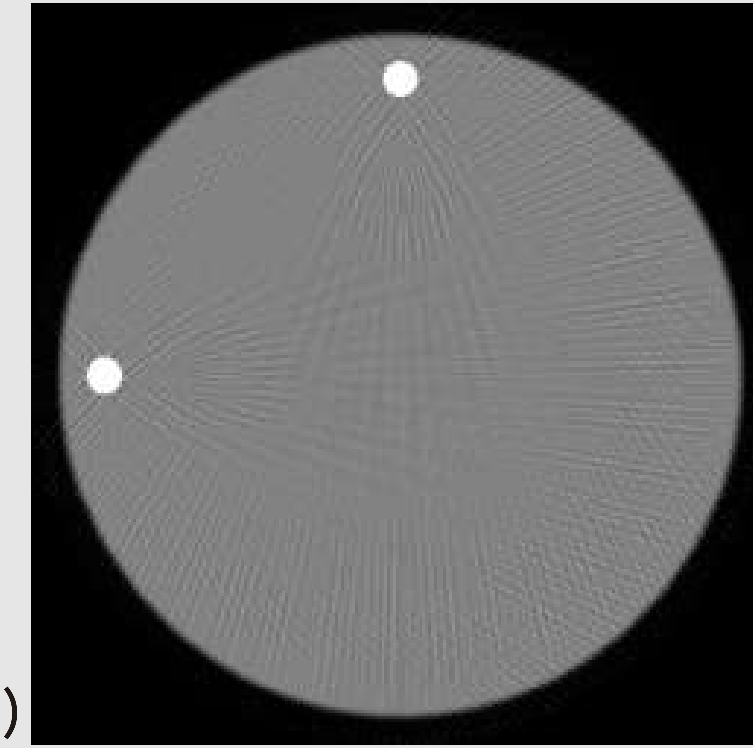
Reconstruction was accomplished using the standard algorithm.

- (a) is the reconstruction using $L_O \cup L_R$
- (b) is a standard reconstruction with half the data and half the detail of (a).

(a)



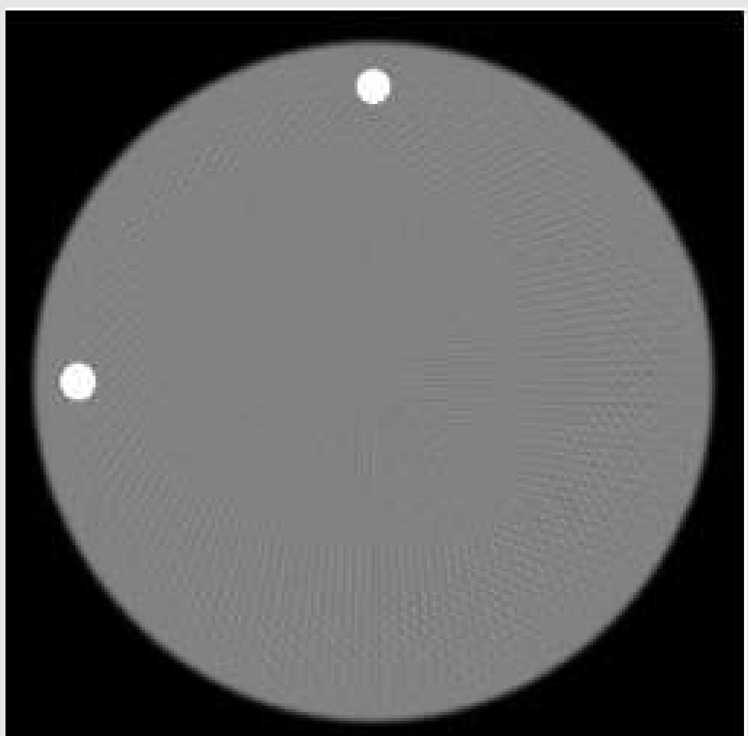
(b)



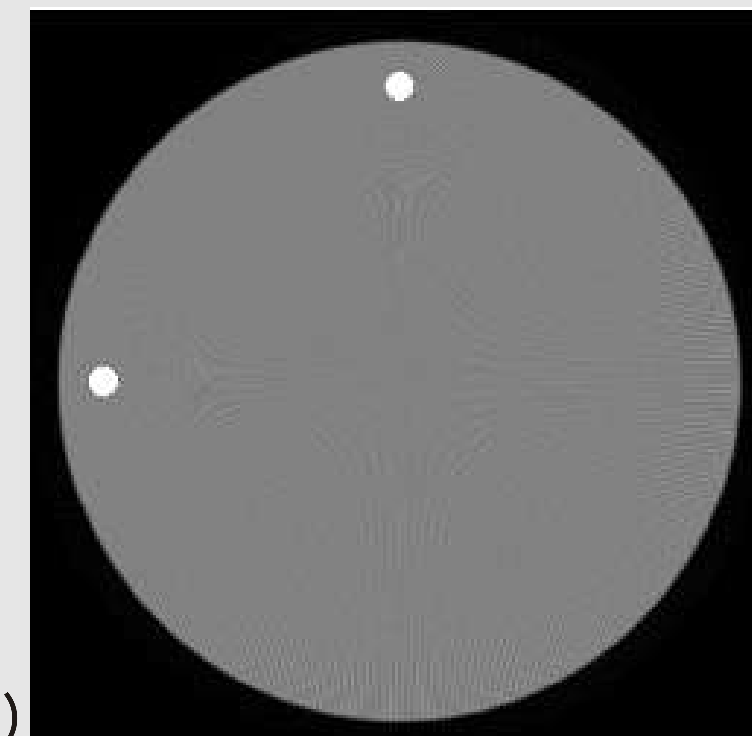
- (c) uses the data of (a) but is no better than (b).

(d) has twice the data of (a) and the level of detail is comparable

(c)



(d)



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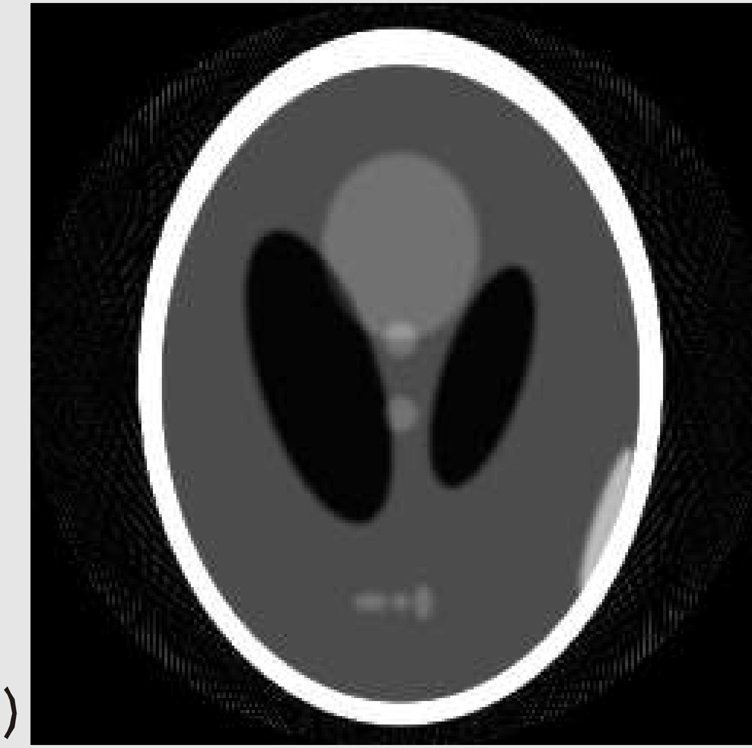
(a)


(b)

- (c) uses the data of (a) but is no better than (b).

(d) has twice the data of (a) and the level of detail is comparable

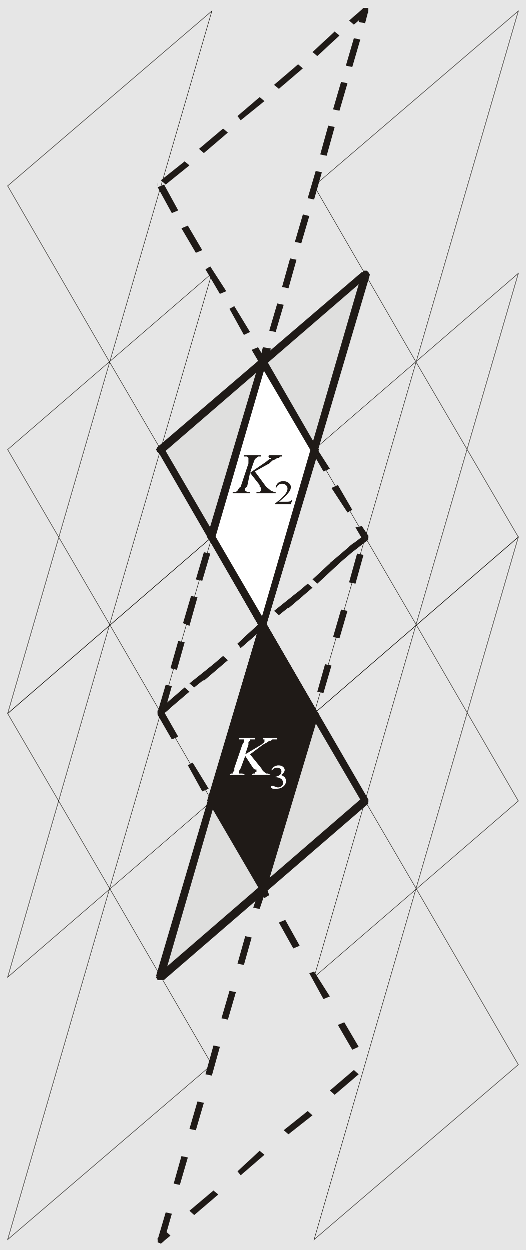

(c)


(d)

K was not Shift-Convex

These reconstructions were done with a bandregion K that was not shift convex.

Thus, the reconstruction algorithm developed by Izen et al. would not have been able to perform the reconstruction using the same lattices we used.



CONCLUSION

Conclusions

- We have verified the results of Izen et al.
- Removed the restrictive hypothesis of shift-convexity
- Illustrated a way to overcome the third-generation problem.

Acknowledgments

- Thanks to Dr. Adel Faridani for his insights and encouragement and for suggesting this problem.
- Special thanks to my wife for her tolerance and patience. Thanks, Lisa!